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BY J. G. COFFIN.

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It is well known, through the work of Lord Rayleigh,[1] and Stefan,[2] that currents of high frequency in straight wires distribute themselves over the cross-section of the wires very differently from steady currents.

The values of the resistance of such wires, as well as their self-inductance, thereby suffer a change. Resistance undergoes a large increase, self-inductance a small decrease. The theory for straight wires has been hitherto directly applied to wires wound on coils, in want of a better theory for such a case.

Recently, Max Wien [3] and A. Sommerfeld [4] have attacked successfully the problem for wire coils.

Max Wien's results, however, are only applicable to small frequencies, while those of Sommerfeld, although valid for all frequencies, gave only the change in *resistance*.

In the following will be found the derivation of a formula by a method based on that of Sommerfeld for the change of self-inductance with the frequency.

By means of this theory we shall derive a method of assigning the correction to be applied to any coil of a single layer. Its extension to coils of more than one layer is reserved for a future communication. The theory [5] shows that with increasing frequency the currents tend to flow more and more on the inside surface of the coil. The maximum change in self-inductance is then the change due to a diminution of the mean radius of the coil of one half the diameter of the wire, with which the coil is wound.

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[1] Phil. Mag., **21**, 381 (1886).
[2] Wied. Ann., **41**, 400 (1890).
[3] Max Wien, Ann. d. Phys., **14**, 1 (1904).
[4] A. Sommerfeld, Ann. d. Phys., **15**, 673 (1904).
[5] Max Wien, loc. cit.

For example, consider a coil whose mean radius is 27 cms. wound with wire of diameter .06 cms. There is a maximum possible change of 3.5 per cent for very high frequency in the self-inductance of a single turn, while in a coil of 716 turns the total possible change is nearly 0.3 per cent. This is not a negligible amount, so that in standard coils and work of precision it is necessary to know what the corrections are for any frequency.

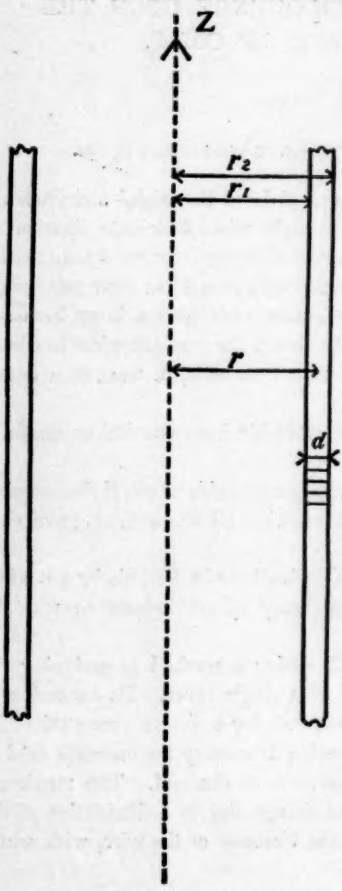


FIGURE 1.

DERIVATION OF THE FORMULA.

Consider an infinite circular current sheet of thickness d, of inside radius r_1 and outside r_2.

Let the symbols $S(\ )$ and $V(\ )$ denote the scalar and vector products respectively. ∇ is the Hamiltonian operator. $V\nabla(\ )$ is the same as curl $(\ )$ or rot $(\ )$, while $S\nabla(\ )$ is the same as div $(\ )$.

By symmetry, the field inside the coil is axial, and a function of r alone, at any given time, and on account of the coil being infinite in length, the field is all inside of it.

In the dielectric, Maxwell's equations are, if we neglect the displacement current,

$$V\nabla H = 0 \text{ and } S\nabla H = 0. \quad (1)$$

From the second relation we find that H is independent of z, and from the first relation, that it is independent of x and y; hence it is constant inside the coil, and for

$$r \gtrless r_1 \text{ assume } H = H_0 e^{int} \quad (2)$$

where ω is $2\pi \times$ frequency, and H_0 the maximum value of H. Outside the coil, H must be constant, but as it vanishes at infinity it is zero everywhere. Hence, for

$$r \gtrless r_2 \qquad H = 0.$$

In the material of the coil, Maxwell's equations are

$$4\pi q = V\nabla H, \quad \frac{dH}{dt} = -V\nabla F, \quad \text{and } q = \sigma F, \tag{3}$$

from which follow $$S\nabla H = 0 \text{ and } S\nabla q = 0 \tag{4}$$

where q is the current density,
F is the electric force,
σ is the specific conductivity.

Eliminating q and F, by taking the curl of the first equation (3)

$$4\pi V\nabla q = V\nabla V\nabla H = \nabla S\nabla H - \nabla^2 H$$
$$= -\nabla^2 H \qquad \text{by (4)}$$

and from the second and third of (3)

$$V\nabla q = \sigma V\nabla F = -\sigma \frac{dH}{dt}$$

therefore $$4\pi\sigma \frac{dH}{dt} = \nabla^2 H. \tag{5}$$

Now assume H in the material of the coil to be given by a function U of r alone (on account of symmetry), multiplied by $H_0 e^{i\omega t}$, or $H = H_0 U e^{i\omega t}$ where

$U = 1$ when $r = r_1$
$U = 0$ " $r = r_2$.

The differential equation then which U must satisfy by equation (5) becomes in polar co-ordinates

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} + k^2 U = 0 \tag{6}$$

where $k = -i\,4\pi\sigma\omega$.

The solutions of this equation are the Bessel's functions of order zero, J_0 and K_0 and hence

$$H = H_0 \{A J_0(kr) + B K_0(kr)\} e^{i\omega t} \tag{7}$$

where A and B are arbitrary constants.

Now, Sommerfeld[6] has shown that the expression

$$H = \sqrt{\frac{r_1}{r}} \frac{e^{ik(r_2-r)} - e^{-ik(r_2-r)}}{e^{ik(r_2-r_1)} - e^{-ik(r_2-r_1)}} H_0 e^{i\omega t} \tag{8}$$

is a very good approximation formula for the case in which kr_1 and kr_2 are large, and he finds also that it is valid even when these quantities are small, this being an unexpected result.

Since

$$4\pi q = V\nabla H = -\frac{\partial H}{\partial r}$$

$$4\pi q = ik\sqrt{\frac{r_1}{r}} \frac{e^{ik(r_2-r)} + e^{-ik(r_2-r)}}{e^{ik(r_2-r_1)} - e^{-ik(r_2-r_1)}} H_0 e^{i\omega t} \tag{9}$$

where in differentiating, on account of the largeness of kr only the exponential terms were considered variable.

For the total current I per unit of length we obtain the value

$$I = \frac{1}{N}\int_{r_1}^{r_2} q\,dr = -\frac{1}{4\pi N}\int_{r_1}^{r_2}\frac{dH}{dr}\,dr = \frac{1}{4\pi N}\{H_{r_1} - H_{r_2}\}$$
$$\therefore I = \frac{H_0}{4\pi N} e^{i\omega t} \tag{10}$$

where N is the number of turns per unit length of the coil.

The electro-kinetic energy of a system is given by

$$T = \frac{1}{2} L I^2 \tag{11}$$

where L is its self-inductance, and I is the total current. In the case of alternating currents, the mean square values are meant, or

$$\tilde{T} = \frac{1}{2} L \tilde{I}^2.$$

Hence we may write

$$L = \frac{2\tilde{T}}{\tilde{I}^2} = 2\,\frac{\frac{1}{\tau}\int_0^\tau T\,dt}{\frac{1}{\tau}\int_0^\tau I^2\,dt} \tag{12}$$

where τ is the period.

[6] Loc. cit., p. 678.

But the mean square value of an harmonic function is one half the square of its amplitude, and in working with imaginary expressions the square of the amplitude can always be obtained by multiplying any expression by its conjugate. Hence

$$\bar{I}^2 = \frac{1}{\tau}\int_0^\tau I^2 dt = \frac{1}{2} I\bar{I} \tag{13}$$

where $\bar{I}$ denotes the conjugate to I.

Therefore

$$I^2 = \frac{1}{2} I\bar{I} = \frac{1}{2}\left(\frac{H_0}{4\pi N}\right)^2$$

and

$$L = 4\left(\frac{4\pi N}{H_0}\right)^2 \frac{1}{\tau}\int_0^\tau T dt. \tag{14}$$

In general,

$$T = \frac{1}{8\pi}\int\int\int_\infty H^2 dv, \tag{15}$$

if μ, the magnetic permeability, is assumed equal to 1 everywhere, which becomes, in this case,

$$T = \frac{1}{8\pi}\int_0^\infty H^2 2\pi r\, dr$$

per unit length of coil

$$\begin{aligned} &= \frac{1}{4}\int_0^{r_1} H^2 r\, dr + \frac{1}{4}\int_{r_1}^{r_2} H^2 r\, dr \\ &= T_1 \qquad\qquad + T_2 \end{aligned} \tag{16}$$

where, in T_1, $H = H_0 e^{i\omega t}$

and in T_2, $H =$ eq. (8).

We may write

$$\frac{1}{\tau}\int_0^\tau T dt = \frac{1}{4}\int r\, dr \frac{1}{\tau}\int_0^\tau H^2 dt = \frac{1}{8}\int H\bar{H} r\, dr.$$

For T_1 this becomes $\left(\frac{H_0 r_1}{4}\right)^2$. (17)

To obtain the integral T_2 first put in equation (8) $h(1+i)$ for ik.

Then, calling $h(r_2 - r) = y$,

and $$h(r_2 - r_1) = y_1,$$

we obtain for H and its conjugate $\bar{H}$, the expressions:

$$H = \sqrt{\frac{r_1}{r}} \frac{e^{iy} e^{y} - e^{-iy} e^{-y}}{e^{iy_1} e^{y_1} - e^{-iy_1} e^{-y_1}} H_0 e^{i\omega t}$$

and (18)

$$\bar{H} = \sqrt{\frac{r_1}{r}} \frac{e^{-iy} e^{y} - e^{iy} e^{-y}}{e^{-iy_1} e^{y_1} - e^{iy_1} e^{-y_1}} H_0 e^{-i\omega t}.$$

A conjugate expression is obtained by putting $-i$ for i everywhere in an equation.

Their product becomes

$$H\bar{H} = H_0^2 \frac{r_1}{r} \frac{(e^{2y} + e^{-2y}) - (e^{2iy} + e^{-2iy})}{(e^{2y_1} + e^{-2y_1}) - (e^{2iy_1} + e^{-2iy_1})}$$

or

$$H\bar{H} = H_0^2 \frac{r_1}{r} \frac{\cosh 2y - \cos 2y}{\cosh 2y_1 - \cos 2y_1}.$$

Multiplying this expression by $r\,dr$ and integrating from r_1 to r_2 gives

as $$\int \cosh r\, dr = \sinh r$$

putting $$x = 2h(r_2 - r_1) = 2hd$$

$$T_2 = \frac{H_0^2}{8} \frac{r_1 d}{x} \frac{\sinh x - \sin x}{\cosh x - \cos x}.$$

We may now write

$$L_\infty = 4\pi^2 N^2 r_1^2 \left[1 + \frac{2}{3} \frac{d}{r_1} \left(\frac{3}{x} \frac{\sinh x - \sin x}{\cosh x - \cos x}\right)\right] \qquad (19)$$

where $x = 2d\sqrt{2\pi\omega\sigma} = (4\pi d\sqrt{\sigma})\sqrt{n}$ where $\omega = 2\pi n$

Now, $4\pi^2 N^2 r_1^2$ is the self-inductance per unit length of a coil with a mean radius r_1, and is therefore the self-inductance for infinite frequency, according to the remarks of Wien. This is also shown by equation (19), as the second term in parenthesis is zero when $x = \infty$ for $\frac{\sinh x}{\cosh x} = 1$, and $\sinh x$ and $\cosh x$ increase without limit as x increases without limit, as may be seen by their expansions

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots; \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots;$$
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots; \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots; \tag{20}$$

Equation (19) may then be written

$$\frac{L_\omega - L_\infty}{L_\infty} = \frac{2}{3}\frac{d}{r_1}\left(\frac{3}{x}\frac{\sinh x - \sin x}{\cosh x - \cos x}\right) = \frac{2}{3}\frac{d}{r_1} Q. \tag{21}$$

By this equation we see that the fractional change of self-inductance in any coil may be calculated if the properties of the function

$$Q = \frac{3}{x}\frac{\sinh x - \sin x}{\cosh x - \cos x} \tag{22}$$

be known for all values of x from zero to infinity.

A formula for small frequencies may be derived by expanding the hyperbolic and circular functions and retaining only low powers of the variable x. Q then becomes

$$9 = 3\,\frac{\frac{1}{3!} + \frac{x^4}{7!} + \frac{x^8}{11!} + \ldots}{\frac{1}{2!} + \frac{x^4}{6!} + \frac{x^8}{10!} + \ldots}$$

which, retaining only terms in x^4 is

$$Q = \frac{\frac{1}{2} + \frac{3x^4}{7!}}{\frac{1}{2} + \frac{6!}{x^4}} = \frac{1 + \frac{6x^4}{7!}}{1 + \frac{2x^4}{6!}} = 1 + \frac{6x^4}{7!} - \frac{14x^4}{7!} = 1 - \frac{8x^4}{7!} = 1 - \frac{x^4}{630}.$$

This equation is applicable for small frequencies only, and (21) may be written, since $64\pi^2 = 630$ very approximately.

$$\frac{L_\omega - L_\infty}{L_\infty} = \frac{2}{3}\frac{d}{r_1}(1 - \omega^2 d^4 \sigma^2). \tag{23}$$

It agrees as to its form and dimensions with that of Max Wien.[7] When $\omega = 0$, it reduces to $2/3\ d/r_1$. As a verification of the term $2/3\ d/r_1$, the approximation formula for Bessel's formula with *small* argument

$$H = H_0 \sqrt{\frac{r_1}{r}}\,\frac{r_2 - r}{r_2 - r_1}\, e^{i\omega t}\,[8] \tag{24}$$

[7] Loc. cit., p. 16.

[8] Sommerfeld, loc. cit., p. 679.

was taken and a calculation similar to the one above for the general case made. This completely verified this term, and *proved* that the expression used above equation (21) is actually valid for *small* arguments as well as large. This last expression $2/3\ d/r_1$ multiplied by $L_\infty = 4\pi^2 N^2 r_1^2$ is

$$L_0 - L_\infty = \frac{8}{3}\pi^2 N^2 r_1 d \qquad (25)$$

and is therefore the additional self-inductance due to the field in the wires themselves for steady values of the current, or, in other words, this is the maximum possible change in self-inductance. Calculations by means of (25) show this expression to give, for coils of *finite* length and wires with *round* instead of square cross-sections, the values:

TABLE I.

COMPARISON OF THE THEORY WITH THE TRUE CHANGE IN SELF-INDUCTANCE.

Coil Length.	True Change.	Value of $2/3\ d/r_1$.	Ratio.
46.0 cms.	.000380 Henry	.000280 Henry	1.35
30.5 "	.000200 "	.000160 "	1.25
15.0 "	.000067 "	.000057 "	1.17
13.0 "	.000051 "	.000044 "	1.16

The column headed "True Change" was computed by taking the difference between the self-inductance of a coil of mean radius a, computed by means of an exact formula, and that of the same coil, using as a mean radius $a - \rho$, where ρ is the radius of the wire. These results, considering the assumptions made in the theory, are in good agreement with it. To make the theory fit to actual facts the second term in the parenthesis in equation (19) should be multiplied by a constant, the average value of which is approximately 1.25, deduced from the above table. We should expect some multiplier to be necessary because a current sheet is the equivalent of square wires without insulation, while, in the coil for which the calculations were made the wires are round and have insulation. We may then write

$$\frac{L_\omega - L_\infty}{L_\infty} = 1.25\,\frac{2}{3}\frac{d}{r_1}\left(\frac{3\sinh x - \sin x}{x\cosh x - \cos x}\right). \qquad (26)$$

In the proposed use of this theory, however, it is not necessary to employ this constant whose value at best is not satisfactorily determinate.

As $\cosh x$ and $\sinh x$ increase without limit as x increases, equation (26) may be written, for large values of ω,

$$\frac{L_\omega - L_\infty}{L_\infty} = 1.25\,\frac{2}{3}\frac{d}{r_1}\frac{3}{x} = \frac{5}{4}\frac{d}{r_1}\frac{1}{d\sqrt{2\pi\omega\sigma}}. \qquad (27)$$

This formula agrees to a constant factor près, with that of Lord Rayleigh,[9] for large values of n for straight wires; i. e.:

$$\frac{L_\omega - L_\infty}{L_\infty} = \sqrt{\frac{R}{2ne}} = 2\frac{1}{d\sqrt{2\pi\omega\sigma}}.$$

TABLE II.

TABLE OF VALUES FOR THE CURVES Q, Q_1, AND Q_2, FOR VALUES OF x FROM ZERO TO INFINITY.

x.	Cosh x.	Sinh x.	Cos x.	Sin x.	Q.	Q_1.	Q_2.
0.0	1.0000	0.0000	1.0000	0.0000	1.0000	1.0000	∞
0.5	1.1276	0.5211	0.8776	0.4789	0.9999	0.9999	6.0000
1.0	1.5431	1.1752	0 5402	0.8415	0.9984	0.9984	3.0000
1.5	2.3523	2.1291	0.0706	0.9975	0.9920	0.9918	2.0000
2.0	3.7622	3.6269	−0.4163	0.9092	0.9756	0.973	1.5000
3.0	10.0677	10.0178	−0.9900	0.1409	0.8932	0.871	1.0000
4.0	27.308	27.262	−0.6534	−0.7570	0.7515	0.594	0.7500
5.0	74.205	74.197	+0.2840	−0.9588	0.6100	Negative	0.6000
6.0	202 approx.	202 approx.	0.9603	−0.2790	0.5032	"	0.5000
7.0	573 approx.	573 approx.	0.7537	+0.6574	0.4287	"	0.4289
8.0	Very large and the same as sinh x.	Very large and the same as cosh x.	....	....	0.3750	"	0.3750
9.0			....	....	0.3333	"	0.3333
10.0			....	....	0.3000	"	0.3000
∞			....	....	0.0000	"	0.0000
∞	∞	∞	Indeterm.	Indeterm.	0	−∞	0

Computations were carefully made for the values of the term

$$Q = \frac{3}{x}\frac{\sinh x - \sin x}{\cosh x - \cos x} \qquad (28)$$

[9] Loc. cit.

for different values of x, as well as of Q_1 and Q_2, where

$$Q_1 = \left(1 - \frac{x^4}{630}\right), \quad \text{and} \tag{29}$$

$$Q_2 = \frac{3}{x}. \tag{30}$$

Table II contains all of the data.

It is clearly seen that as long as $x < 2$ that formula Q_1 may be used, while if $x > 7$, formula Q_2 is valid. In between these points, the calculated values will serve.

A plot of these three functions on the next page shows still more clearly the relations of the three formulae for different values of x.

Manner of Using the Curve.

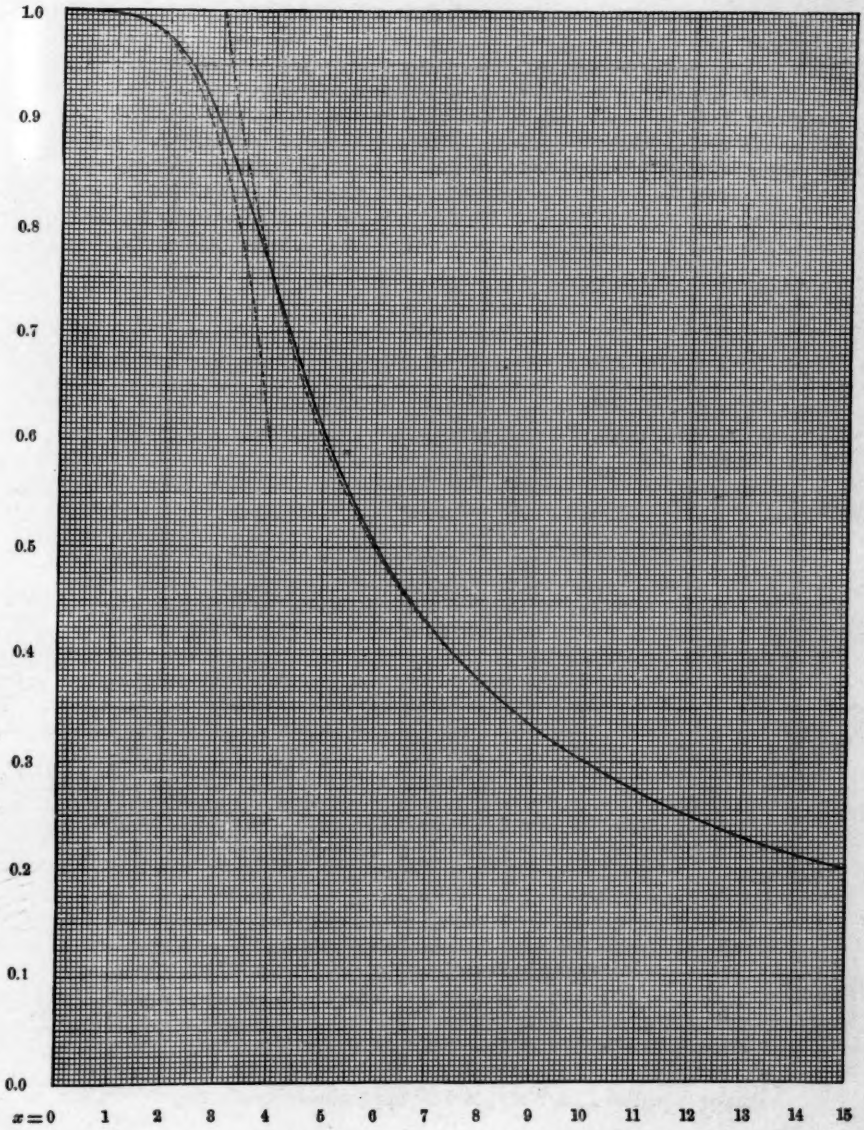
This curve is to be used in the following manner. We assume that in any coil of a single layer, the falling off of the self-inductance takes place according to equation (22), represented by the full-line curve, and the maximum ordinate unity is taken to be the maximum change possible of the self-inductance, which is otherwise calculable. For any given dimensions of the coil, the values of x correspond to definite frequencies, and either a table or a plot of corrections may be derived from the above curve.

For small values of x, and therefore for small values of the frequency, the correction varies proportionately to d^4, while for very large frequencies it is proportionate to $1/d$ by equations (29) and (30).

As the first case is the more usual, it is important, in order to avoid the necessity of any correction, to wind the coil with wires of small diameter. The size necessary to avoid a correction under a given frequency may easily be calculated. As an application of this theory, the following example will be carried out:

Consider a coil of 27 cms. radius $= r_1$ wound with a single layer of wire whose diameter d is .063 cms. Let the conductivity of the wire be taken as 5.9×10^{-4}. Then the relation between x and n is $n = 2630\, x^2$ from (19) where $n =$ frequency. In the following table, column 2, are the values of x corresponding to the frequencies n in column 1. In column 3 are the values of the ordinates corresponding to these values of x taken from the Plot (I).

PLOT I.



Plot of the functions $Q = \frac{6}{x}\frac{\sinh x - \sin x}{\cosh x - \cos x}$ full line.

$Q_1 = (1 - x^4/630)$ dotted line.

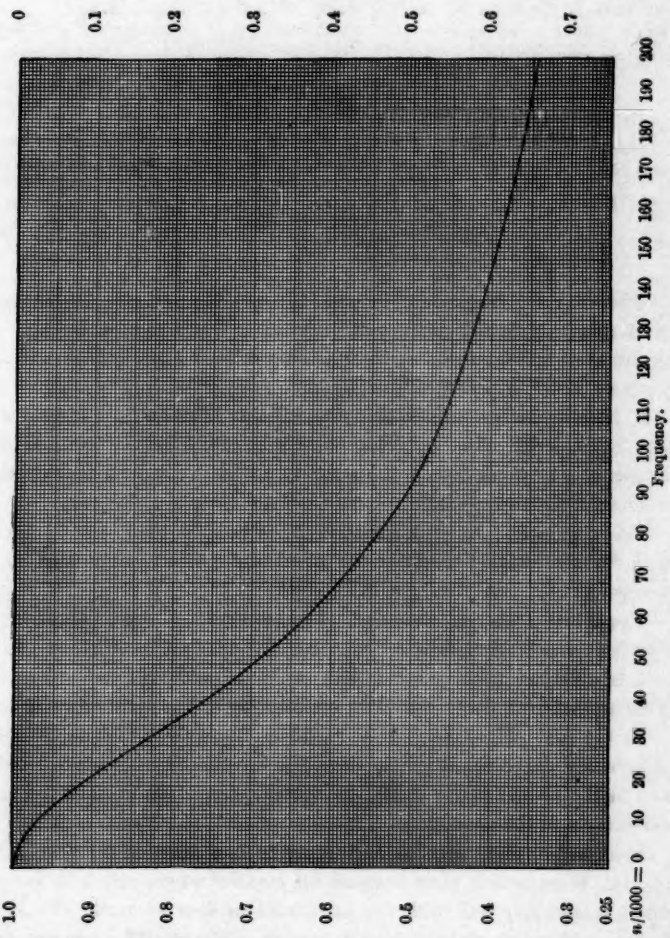
$Q_2 = 3/x$ dotted line.

TABLE III.

n	x	Ordinates.
0	0	1.000
500	0.436	1.000
1000	0.616	0.999
1000	0.755	0.998
3000	1.068	0.996
4000	1.230	0.995
5000	1.380	0.993
6000	1.518	0.990
10000	1.950	0.977
15000	2.380	0.950
20000	2.750	6.916
30000	3.380	0.846
40000	3.900	8.765
50000	4.350	8.700
60000	4.780	0.638
70000	5.150	0.590
80000	5.510	0.550
90000	5.850	0.515
100000	6.170	0.483
120000	6.760	0.463
150000	7.550	0.397
170000	8.050	0.373
200000	8.720	0.344

The plot of the corresponding value of n as abscissae and values of column 3 as ordinates gives a curve of corrections for the above coil. For example, at a frequency of 50,000 there is 0.7 of the total possible change (equal to L_0 2/3 $d/r_1 \times 1.25$ approximately) still to go. Or, the

PLOT II.



Curve showing the fractional part of the maximum decrease for different frequencies in a coil, the wire of which has a diameter of .063 cm.

Ordinates = fraction of total change left unchanged.

$$\text{Abscissae} = \frac{\text{frequencies}}{1000}.$$

For example, at 30,000 $\sim$ per second the correction is 0.157×0.0000618 Henrys in 0.0362 Henrys, or about 1 part in 3600.

correction is 0.3 of the maximum change. The value of the correction may be best obtained by reading *down* from the ordinate unity.

These computations also show that equation (29)

$$q_1 = 1 - \frac{x^4}{630}$$

may be employed up to frequencies of 10,000 for this particular coil, and that equation (30)

$$q_2 = \frac{3}{x}$$

may be used for frequencies beyond 120,000 per second. As the maximum correction is 0.2 per cent of the total, if frequencies under 4000 per second are used, the error will not be over 0.2 per cent $\times$ 0.005 = 0.001 per cent, which is practically negligible.

If, however, other things being equal, the diameter of the wire were four times larger than it is, the same corrections would correspond to frequencies 16 times less, or an error of 0.001 per cent is now obtained at a frequency of only 250.

This suggests, if large currents must be carried, that to wind the coil with flat wires would diminish the effect of the frequency upon both the changes in self-inductance and resistance.

The Plots I and II, just derived, show how the self-inductance of a circular *current sheet* falls off with the frequency. In other words, the coil is assumed to be wound with wires of rectangular cross-section. This may cause a doubt as to the applicability of the curve in question.

In the first place, the method of its use does away with the necessity of knowing the constant multiplier of the function Q, as explained above; and in the second place, the following two reasons go to show the entire correctness of the deductions.

Sommerfeld finds that his results agree to a constant près with those of Max Wien, which were deduced for *circular* wires, and both formulae agree remarkably well with the experimental data at hand; this for the increase of resistance with the frequency. We should, therefore, fairly expect the results for the decrease in self-inductance to also be correct to within a constant factor. As there are no data available for the decrease in self-inductance, we have not been able to verify this conclusion. To further assure ourselves, however, we have worked out the problem of the change of self-inductance with the frequency in an infinite straight circular-ring conductor, using the approximation formulae for the Bessel's

functions as above given. It was found that the *variable part* of the expression for self-inductance agreed *identically* with that derived for a circular current sheet. This shows that the bending of straight wires into circles, and forming a coil of a single layer therewith, does not affect the *way* in which the self-inductance varies.

We believe, then, that Plot I gives very approximately the law according to which the self-inductance falls off for straight ring-shaped conductors, and probably for any simply shaped square or polygonal hollow straight conductors, as well as for circular coils of a single layer, wound with wires of circular, square, or polygonal cross-sections.

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